

The Conformal Properties of Liouville Field Theory on \mathbb{Z}_N -Riemann Surfaces

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Abstract

The Liouville field theory on \mathbb{Z}_N -Riemann surfaces is studied and it is shown that it decomposes into a Liouville field theory on the sphere and $N - 1$ free boson theories. Also, the partition function of the Liouville field theory on the \mathbb{Z}_N -Riemann surfaces is expressed as a product of the correlation function for the Liouville vertex operators on the sphere and a number of twisted fields.

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1 Introduction

The main motivation to study two-dimensional Liouville field theory (LFT) is its relation to the string theories and the realization that it provides an effective theory of 2D quantum gravity. However, despite significant progress in understanding the classical Liouville theory, our understanding of the quantum Liouville field theory is quite limited. However, since the 80's, essential progress has been achieved in the understanding of quantum LFT ([1, 2] and references therein). The interest on LTM was intensified with the development of the matrix model approach that confirmed results obtained with the LFT ([3] and references therein). Recently, the interest on LFT has again been renewed since an analytic expression for the three-point correlation function of the Liouville vertex operators has been constructed [4, 5, 6].

The present work is organized as follows. In the first part of section 2, a brief description of the LFT on \mathbb{Z}_N -surfaces is given. In the second part of section 2, we use Polyakov's proposal, to express the partition function of the LFT on a \mathbb{Z}_N -surface as a partition function of a LFT on a sphere and free scalar field theories with inserted Liouville vertex operators and twisted fields. The present paper generalizes a previous work of one of the authors [15].

2 LFT on \mathbb{Z}_N -Riemann surfaces.

A \mathbb{Z}_N -symmetric Riemann surface $X_g^{(N)}$ of genus $g \geq 1$ is determined by an algebraic equation of the form

$$y^N(z) = \prod_{i=1}^h (z - \omega_i)^{n_i} , \quad n_i > 1 ,$$

i.e. $X_g^{(N)}$ -is an N -sheeted covering of a Riemann sphere. The genus g of a \mathbb{Z}_N -Riemann surface

$$g = \frac{(N-1)(h-2)}{2}$$

can be calculated using the Riemann-Hurwitz theorem. The algebraic equation of the \mathbb{Z}_N -surface has h complex parameters; therefore the moduli space

$\mathcal{M}_{\mathbb{Z}_N}$ has dimension

$$\dim \mathcal{M}_{\mathbb{Z}_N} = h - 3 = \frac{2g}{N-1} - 1 .$$

Comparing with the dimension of the moduli space \mathcal{M} for generic Riemann surfaces,

$$\dim \mathcal{M} = \begin{cases} 1, & \text{if } g = 1 , \\ 3g - 3, & \text{if } g > 1 , \end{cases}$$

we can conclude that the \mathbb{Z}_N -surfaces do not contain all Riemann surfaces.

We start with Liouville theory in the conformal gauge. The action is given by

$$\mathcal{S} = \int_{\mathbb{Z}_N} d^2y \left[\frac{1}{4\pi} (\partial_a \phi)^2 + \mu e^{2b\phi} + \frac{Q}{4\pi} \hat{R} \phi \right] ,$$

where b and μ are the coupling and cosmological constants respectively. We have fixed a fiducial metric \hat{g} on a given surface with curvature \hat{R} normalized by

$$\frac{1}{4\pi} \int_{\mathbb{Z}_N} d^2y \sqrt{\hat{g}} \hat{R} = 2(1 - g) .$$

We label the N sheets of the Riemann \mathbb{Z}_N -surface $X_g^{(N)}$ by the numbers $l = 0, 1, \dots, N-1$:

$$y^{(l)}(z) = \omega^l \prod_{j=1}^h (z - \omega_j)^{n_j/N} . \quad (1)$$

Under the map (1), the Lagrangian density $\mathcal{L}(\phi(y))$, the energy-momentum tensor $T(\phi(y))$, and the Liouville fields $\phi(y)$ on the \mathbb{Z}_N -Riemann surface map into branches: $\mathcal{L}^{(l)}(\phi^{(l)}(z))$, $T^{(l)}(z)$, and $\phi^{(l)}(z)$, $l = 0, 1, \dots, N-1$ respectively.

Let $\Omega(z)$ be the conformal factor of the metric under conformal transformations of the coordinates. The Liouville branch fields $\phi^{(l)}(z)$ transform like a logarithm of the conformal factor:

$$\phi^{(l)}(\omega, \bar{\omega}) = \phi^{(l)}(z, \bar{z}) - \frac{Q}{2} \log |\Omega'(z)|^2 ,$$

where

$$Q = b + \frac{1}{b} .$$

On each sheet, we have a holomorphic Liouville energy-momentum tensors

$$T^{(l)}(z) = -(\partial \phi^{(l)}(z, \bar{z}))^2 + Q \partial^2 \phi^{(l)}(z, \bar{z}) ,$$

with the Liouville central charge

$$\hat{c} = 1 + 6Q^2 .$$

As usual, we assume that the fields $(T^{(l)}(z), \phi^{(l)}(z, \bar{z}))$ on the N -sheeted covering may be considered as vector fields on \mathbb{CP}^1 . When the argument of these vector fields encircles the branch points, they transform among themselves according to a certain monodromy matrix. This monodromy matrix forms a representation of the first homotopy group $\pi_1(\mathbb{CP}^1 / \cup \omega_j)$ which in our case is just \mathbb{Z}_N . It is convenient to pass to a basis [9] in which the generators of monodromy group are diagonal:

$$\phi_{(k)}(z, \bar{z}) = \sum_{l=0}^{N-1} \omega^{-kl} \phi^{(l)}(z, \bar{z}) ,$$

$$T_{(k)}(z) = \sum_{l=0}^{N-1} \omega^{-kl} T^{(l)}(z) .$$

The “bosonization rule” for the operators $T_{(k)}$ in the diagonal basis can be written as follows:

$$T_{(k)} = -\frac{1}{N} \sum_{s=0}^{N-1} \partial \phi_{(s)} \partial \phi_{(k-s)} + Q \partial^2 \phi_{(k)} .$$

In particular, the form of the Liouville energy-momentum tensor is given by

$$T_{(0)} = -\frac{1}{N} \partial \phi_{(0)} \partial \phi_{(0)} + Q \partial^2 \phi_{(0)} - \frac{1}{N} \sum_{s \neq 0}^{N-1} \partial \phi_{(s)} \partial \phi_{(-s)} \quad (2)$$

and the corresponding Liouville central charge is

$$c = N (1 + 6Q^2) .$$

According to the previous results, the original theory splits into a sum of a LFT on a sphere with the central charge $c_s = 1 + 6Q^2 N$ and $N - 1$ free field theories $\phi_{(s)}$, $s = 1, 2, \dots, N - 1$ with central charge $c_f = 1$.

Under a holomorphic transformation of the coordinates, the Liouville field $\phi_{(0)} \equiv \Phi$ and the free fields $\phi_{(k)}$ ($k \neq 0$) transform as follows:

$$\Phi(\omega, \bar{\omega}) = \Phi(z, \bar{z}) - \frac{N}{2} Q \log |\Omega'(z)|^2 ,$$

$$\phi_{(k)}(\omega, \bar{\omega}) = \phi_{(k)}(z, \bar{z}) , \quad k \neq 0 .$$

We rewrite (2) in terms of new variables:

$$T = -\frac{1}{N}\partial\Phi\partial\Phi + Q\partial^2\Phi - \frac{1}{N}\sum_{s \neq 0}^{N-1} \partial\phi_{(s)}\partial\phi_{(-s)} .$$

According to the monodromy properties [9] of the vector fields on \mathbb{CP}^1 , we have to define two kinds of “Liouville vertex operators”. The first kind is “untwisted vertex operators”:

$$V_{[0]}(z, \bar{z}) = e^{2\alpha\Phi(z, \bar{z})} e^{i\sum_{s \neq 0} \alpha_{(s)}\phi_{(s)}(z, \bar{z})} .$$

with dimensions

$$\Delta_{[0]} = 2\alpha(Q - \alpha) + \frac{N}{2}\sum_{s \neq 0} \alpha_{(s)}\alpha_{(-s)} . \quad (3)$$

The physical LFT space of states is defined by the following charge [10, 11]:

$$\alpha = ip + \frac{Q}{2} .$$

Substituting this value in the expression (3), we find

$$\Delta_{[0]} = \frac{Q^2}{2} + 2p^2 + \frac{N}{2}\sum_{s \neq 0} \alpha_{(s)}\alpha_{(-s)} .$$

The second kind of vertex operators is the “twisted vertex operators” which have the form:

$$V_{[k]}(z, \bar{z}) = e^{2\gamma\Phi(z, \bar{z})} \sigma_k(z|1)\sigma_k(z|2)\dots\sigma_k(z|N-1) .$$

In the above formula, $\sigma_k(z|l)$ is a twist fields having dimension

$$\Delta_{kl} = \frac{1}{4}[\{\frac{kl}{N}\} - \{\frac{kl}{N}\}^2] ,$$

where the symbol $\{x\}$ denotes the fractional part of x . Thus, the twisted vertex operators have dimensions:

$$\Delta_{[k]} = 2\gamma(Q - \gamma) + \frac{N^2 - 1}{24N} .$$

We now proceed to construct the partition function of the LFT on the \mathbb{Z}_N -surface $X_g^{(N)}$ by making use of the above results. According to a main proposal of Polyakov [8], a “summation” over a smooth metric with insertion of vertex operators at some points should be equivalent to the “summation” over a metric with singularities at the insertion points and without insertion of any vertex operators. Therefore, the partition function of the Liouville field theory on the \mathbb{Z}_N -surface,

$$Z_g = \int D\phi \exp \left[- \int \frac{1}{4\pi} (\partial_a \phi)^2 + \mu e^{2b\phi} + \frac{Q}{4\pi} \hat{R}^g \phi \right] ,$$

can be represented by the expression

$$\begin{aligned} Z_g &= \int D\Phi \exp \left[- \int \frac{1}{4\pi N} (\partial_a \Phi)^2 + \mu e^{2b\Phi} + \frac{Q}{4\pi} \hat{R}^{g=0} \Phi \right] \\ &\times \prod_{s \neq 0} D\phi_{(s)} \exp \left[- \int \frac{1}{4\pi N} \sum_{s \neq 0} \partial \phi_{(s)} \partial \phi_{(-s)} \right] \\ &\times \prod_{i=1}^h e^{2\gamma_i \Phi(\omega_i, \bar{\omega}_i)} \sigma_{k_i}(\omega_i|1) \sigma_{k_i}(\omega_i|2) \dots \sigma_{k_i}(\omega_i|N-1) , \end{aligned} \quad (4)$$

where

$$b = \frac{Q}{2} \pm \sqrt{\frac{Q^2}{4} - \frac{1}{2}} .$$

In order to evaluate the partition function written above, we will first integrate over the zero mode of Φ . After this integration is performed, we find

$$\begin{aligned} Z_g &= (-\mu)^s \frac{\Gamma(-s)}{b} \\ &\times \int D\tilde{\Phi} \exp \left[- \int \frac{1}{4\pi N} (\partial \tilde{\Phi})^2 + \frac{Q}{4\pi} \hat{R}^{g=0} \tilde{\Phi} \right] \left(\int e^{2b\tilde{\Phi}} \right)^s \prod_{i=1}^h e^{2\gamma_i \tilde{\Phi}(\omega_i, \bar{\omega}_i)} \\ &\times \int \prod_{s \neq 0} D\phi_{(s)} \exp \left[- \int \frac{1}{4\pi N} \sum_{s \neq 0} \partial \phi_{(s)} \partial \phi_{(-s)} \right] \\ &\times \prod_{i=1}^h \sigma_{k_i}(\omega_i|1) \sigma_{k_i}(\omega_i|2) \dots \sigma_{k_i}(\omega_i|N-1) , \end{aligned} \quad (5)$$

where

$$\sum_{i=1}^h \gamma_i = Q - sb , \quad (6)$$

and $\tilde{\Phi}$ denotes fields orthogonal to the zero mode [7]. The correlation function in (4) of the fields σ_k is determined by:

$$\langle \prod_{i=1}^h \sigma_{k_i}(\omega_i|1) \sigma_{k_i}(\omega_i|2) \dots \sigma_{k_i}(\omega_i|N-1) \rangle = \prod |\omega_i - \omega_j|^{-2\gamma_{ij}} (\det \widehat{W})^{-1/2} ,$$

where the matrix \widehat{W} is the period matrix and γ_{ij} forms a basis in $H_1(X_g^{(N)}, \mathbb{Z})$ [12]. Moreover, it is well-known that the multi-point correlator (5) leads to the partition function of the free scalar fields under the Ramond boundary condition [13]. The first correlation function in (4) is not a free field correlator because, in general, the power s is not a positive integer. However, for integer values, $s = n \in \mathbb{Z}$, the partition function Z_g exhibits a pole in the $\sum \gamma_i$ with the residue being equal to the corresponding perturbative integral

$$\text{Res}_{\sum \gamma_i = Q - nb} Z_g(\omega_1, \dots, \omega_h) = G^{(n)}(\omega_1, \dots, \omega_h) \Big|_{\sum \gamma_i = Q - nb} ,$$

where $G^{(n)}$ is the free field correlator

$$\begin{aligned} G^{(n)} &= \frac{(-\mu)^n}{n!} \int D\Phi e^{-\int \frac{1}{4\pi N} (\partial\Phi)^2 + Q \hat{R}^{g=0} \Phi} \\ &\times \prod_{j=1}^n \int \sqrt{\hat{g}} e^{2b\Phi(x_j)} d^2x_j \prod_{i=1}^h e^{2\gamma_i \Phi(\omega_i, \bar{\omega}_i)} \\ &\times \int \prod_{s \neq 0} D\phi_{(s)} \exp \left[-\frac{1}{4\pi N} \int \sum_{s \neq 0} \partial\phi_{(s)} \partial\phi_{(-s)} \right] \\ &\times \prod_{i=1}^h \sigma_{k_i}(\omega_i|1) \sigma_{k_i}(\omega_i|2) \dots \sigma_{k_i}(\omega_i|N-1) . \end{aligned}$$

This is just the n -th term in the naive perturbation of Z_g in powers of μ . So, the LFT partition function on the \mathbb{Z}_N -surface has been reduced to the Liouville correlation function on the sphere with inserted Liouville vertex operators (with charges γ_i) and to a correlation function of the twisted fields

[14]. The residue of the LFT partition function on the \mathbb{Z}_N -surface at the poles are the correlation functions of the free field theories on the \mathbb{Z}_N -surface.

Let us consider the special case of the Liouville field theory on an elliptic curve, i.e. $N = 2, h = 4$. We can rewrite the partition function Z_1 for this particular case as:

$$\begin{aligned} Z_1 &= \int D\Phi \exp \left[- \int \frac{1}{8\pi N} (\partial_a \Phi)^2 + \mu e^{2b\Phi} + \frac{Q}{4\pi} \hat{R}^{g=0} \Phi \right] \\ &\times \int D\phi \exp \left[- \int \frac{1}{8\pi} \partial\phi \partial\phi \right] \prod_{i=1}^4 e^{2\gamma_i \Phi(\omega_i, \bar{\omega}_i)} \sigma_{\epsilon_i}(\omega_i, \bar{\omega}_i) . \end{aligned}$$

The residue of Z_1 (corresponding to (6)) will be equal to the conformal four-point function

$$\text{Res}_{\sum \gamma_i = Q - nb} Z_1(\omega_1, \dots, \omega_4) = G^{(n)}(\omega_1, \dots, \omega_4) \Big|_{\sum \gamma_i = Q - nb} ,$$

where $G^{(n)}$ has the form

$$\begin{aligned} G_{\gamma_1 \gamma_2 \gamma_3 \gamma_4}^{(n)}(\omega_1, \omega_2, \omega_3, \omega_4) &= \frac{(-\mu)^n}{n!} \int D\Phi e^{- \int \frac{1}{8\pi} (\partial\Phi)^2 + Q \hat{R}^{g=0} \Phi} \\ &\times \prod_{j=1}^n \int e^{2b\Phi(x_j)} d^2 x_j \prod_{i=1}^4 e^{2\gamma_i \Phi(\omega_i, \bar{\omega}_i)} \int_{\phi \in S^1} D\phi \exp \left[- \frac{1}{8\pi} \int \partial\phi \partial\phi \right] . \end{aligned}$$

We can thus conclude that the Liouville partition function on an elliptic curve reduces to the four-point correlation function of the Liouville vertex operators on the sphere and the partition function of the free field theory where integration goes over the compactified fields with Ramond boundary conditions.

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